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DEPARTMENT OF MATHEMATICS ANN

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This seems to be the only counting result for AV's

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Dear Daniel,

While writing up my Durham talk I seemed to get the following slightly improved bounds for degrees and heights. Or am I overlooking something?

Let A be an abelian variety of dimension g, defined over a number field k and embedded in projective space. Let c's be positive constants depending only on A and k (in fact only on A and the degree of k). Let q be the absolute Néron-Tate height on $A(\bar{k})$.

<u>Theorem.</u> There exists c such that for any extension K of k of degree at most D we have

$$\#\{P \text{ in } A(K) : q(P) \leq c^{-1}D^{-1}\} \leq cD^g(\log D)^g.$$

Note that P is <u>not</u> assumed non-torsion, or even non-zero. So we deduce

<u>Corollary 1.</u> $\#A(K)_{\text{torsion}} \leq cD^g(\log D)^g$

Since the same bound holds for the <u>exponent</u> of $A(K)_{\text{torsion}}$, we get

$$d(e) \gg (n(e))^{(1/g)-\varepsilon}$$

in the notation of your Australian note (and this is Paula's result for g = 1).

A standard argument on integer multiples of P gives also

<u>Corollary 2.</u> $q(P) \geq c^{-1}D^{-(2g+1)}(\log D)^{-2g}$ for all P in $A(K)_{\text{nontorsion}}$.

This greatly improves the Anderson-Masser exponent of 10 for g = 1. If A has CM then taking the full endomorphism ring gives $q >> D^{-2}(\log D)^{-1}$.

Taking $K = k(A_n)$ (points of order dividing n) gives

<u>Corollary 3.</u> $[k(A_n):k] \geq c^{-1}n^2/\log n$.

For g = 1 this is my old Bull. London Math. Soc. result.

Here is a sketch of the proof (I can't really say that there are any new ideas - indeed I'm only writing this out as a method of detecting mistakes). Take a large constant C, and use r for the "norm squared" on $A(\mathbb{C})$ induced by some norm on $\mathbb{C}^g$ (e.g. the sup norm)

<u>Step 1.</u> (standard). We can assume A is simple.

<u>Step 2.</u> (standard). Use the Box Principle to show that it suffices to deduce a contradiction from the existence of a finite subset $\mathcal{S}$ of A(K) whose points P satisfy

$$q(P) \leq C^{-4}D^{-1}, \quad r(P) \leq C^{-4}$$

and

$$S = \#\mathcal{S} = C^{4g+1}D^g(\log D)^g.$$

Step 3 (a new remark?). As $\mathcal{S}$ has no "additive structure" we use the set $\mathcal{S}^{(g)} = \{P_1+\ldots+P_g\colon P_1,\ldots,P_g \text{ in } \mathcal{S}\}$. For P in $\mathcal{S}^{(g)}$ we have

$$q(P) \le X^{-1}, \qquad X = g^{-2}C^4D \tag{1}$$

$$r(P) \le Y^{-1}, \qquad Y = g^{-2}C^4 \tag{2}$$

Thus $\mathcal{S}^{(g)}$ has properties like those of $\mathcal{S}$; but it is much better for doing zero estimates.

Step 4. (a minor technical point, but amusing). Write down a set B of basis elements β of K over $\mathbb{Q}$ with

$$\log \text{height}(\beta;\ \beta \text{ in } B) \ll D. \tag{3}$$

Note that we don't have a bounded number of generators for K over $\mathbb{Q}$; for positive integers $d_1,\ldots,d_n$ the inequality

$$\sum_{i=1}^{n} (d_i-1) \le \left(\prod_{i=1}^{n} d_i\right)-1$$

is useful.

Step 5 (Schwarz → Waldschmidt). Since we don't know a Schwarz Lemma for $\mathcal{S}$ (or $\mathcal{S}^{(g)}$) we have to use Waldschmidt's "Théorème 3.1" of his Inventiones paper. Take his L as

$$L_W = C^{6g}D^{2g+1},$$

his S as

$$S_W = C^{-g+4}D,$$

his U as

$$U = C^4D^2,$$

and

$$r = 2Y^{-\frac{1}{2}}, \qquad R = er.$$

The functions are the

$$\varphi_\lambda = \beta\Theta^{\underline{\lambda}_1}(\underline{z})\Theta^{\underline{\lambda}_2}(N\underline{z}) \qquad (\beta \text{ in } B, |\underline{\lambda}_1| = |\underline{\lambda}_2| = L) \qquad (4)$$

with our L as

$$L = c^3D$$

and

$$N = c^2D^{\frac{1}{2}}.$$

Here $\Theta^{\underline{\lambda}} = \vartheta_0^{\lambda_0}\cdots\vartheta_g^{\lambda_g}$ for $\underline{\lambda} = (\lambda_0,\ldots,\lambda_g)$ and homogeneously algebraically independent theta-functions. Check that L_w, the number of functions φ_λ, is equal to DL^{2g}, at least up to constants.

There are two conditions yet to verify. One is

$$(8U)^{g+1} \leq L_wS_w(\log R/r)^g$$

(immediate). The second is

$$\sum_\lambda|\varphi_\lambda|_R \leq e^U.$$

But

$$\log |\varphi_\lambda|_R \ll D^2 + L + LN^2R^2 \qquad (5)$$

and this gives it.

The conclusion is that there are rational integers p_λ with

$$0 < \max_\lambda |p_\lambda| \leq e^{S_w} \qquad (6)$$

and

$$F = \sum_\lambda p_\lambda\varphi_\lambda$$

satisfying

$$|F|_r \leq e^{-U}.$$

Step 6 (standard). Use Cauchy to deduce that

$$|\Delta F|_{\frac{1}{2}r} \leq e^{-\frac{1}{2}U}$$

for all $\Delta = (\partial/\partial z_1)^{t_1}\ldots(\partial/\partial z_g)^{t_g}$ of order at most

$$T = c^3D/\log D.$$

In particular

$$|\Delta F(\underline{u})| \leq e^{-\frac{1}{2}U} \qquad (7)$$

for each $\underline{u}$ in $\mathbb{C}^g$ with $|\underline{u}| \leq Y^{-\frac{1}{2}}$ that corresponds to some P in $\mathcal{S}^{(g)}$ (see (2)).

Step 7 (zero estimate). There exists P_0 in $\mathcal{S}^{(g)}$ and Δ_0 with $|\Delta_0| < T$ such that

$$\Delta_0 F(\underline{u}_0) \neq 0 \qquad (8)$$

for the corresponding $\underline{u}_0$.

We do this as is sketched at the end of §3(a) of your Australian note. Consider F as a function on A of degree $<< LN^2$; it is of course not zero because $N^2 \geq CL$. In view of Wüstholz's generalized "$\omega_{t,A} >> t\omega_{1,A}$" estimates, it suffices to have sharp zero estimates for the set $\mathcal{S}^{(g)}$; but these are given in ZEGV II (quite possibly the whole thing, multiplicities included, could be done by Philippon, but I didn't check this). In fact since A is simple, we get easily

$$\omega_{1,A}(\mathcal{S}^{(g)}) >> (\#\mathcal{S})^{1/g} = S^{1/g}.$$

The required assertion (8) follows from comparing LN^2 and

$$TS^{1/g} \geq C^{1/g}LN^2.$$

Henceforth assume that Δ_0 is minimal for this P_0.

Step 8 (standard). Let f be F divided by a suitable theta function Θ Then

$$\Delta_0 f(\underline{u}_0) = \Theta(\underline{u}_0)^{-1}\Delta_0 F(\underline{u}_0)$$

Lower bounds for Θ are purely technical and just follow the calculation (5). We find from (7)

$$|\Delta_0 f(\underline{u}_0)| < e^{-\frac{1}{4}U} \qquad (9)$$

Finally we have to estimate the height. From (3),(4) and (6) we find the logarithmic bound

$$\ll S_w + D + T\log T + T\log N + (T + L)(1 + q(NP_0)).$$

By (1) this is at most $\ll C^3D$. Thus we get

$$\log |\Delta_0 f(\underline{u}_0)| \gg -C^3D^2,$$

contradicting (9).

This completes the sketch of the proof. I can't see any mistakes; but you know how it is with these things.

Best wishes,

David

P.S. For g=1 I am fairly sure it's OK; I even wrote out a detailed proof with the dependence on the logarithmic height h of g_2, g_3 worked out. It gives for example

$$[k(E_n):k] \gg h^{-\frac{1}{2}}n^2(h + \log n)^{-1}$$

for Corollary 3, with an implied constant depending only on k.

I intend also to work out the dependence on h = h(A) for g > 1; but this won't be quite as straightforward.

P.P.S. It is nice that the points P of A(K) which are counted in the Theorem are those with $q \ll D^{-1}$ - the opposite inequality here isjust the analogue of Lehmer's problem, of course.

no: it is $q \gg D^{-1/g}$